

Capturing Forms in Dense Subsets of Finite Fields

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Abstract

An open problem of arithmetic Ramsey theory asks if given a finite r -colouring $c : \mathbb{N} \rightarrow \{1, \dots, r\}$ of the natural numbers, there exist $x, y \in \mathbb{N}$ such that $c(xy) = c(x + y)$ apart from the trivial solution $x = y = 2$. More generally, one could replace $x + y$ with a binary linear form and xy with a binary quadratic form. In this paper we examine the analogous problem in a finite field \mathbb{F}_q . Specifically, given a linear form L and a quadratic form Q in two variables, we provide estimates on the necessary size of $A \subset \mathbb{F}_q$ to guarantee that $L(x, y)$ and $Q(x, y)$ are elements of A for some $x, y \in \mathbb{F}_q$.

1 Introduction

In this paper we consider a finite field analog of the following open problem in arithmetic Ramsey theory [3].

Problem: For any r -colouring $c : \mathbb{N} \rightarrow \{1, \dots, r\}$ of the natural numbers, is it possible to solve $c(x + y) = c(xy)$ apart from the trivial solution $(x, y) = (2, 2)$?

One might suspect that in fact a stronger result might hold, namely that any sufficiently dense set of natural numbers contains the elements $x + y$ and xy for some x and y . This

*The author is supported in part by NSERC of Canada.

would immediately solve the problem since one of the colours in any finite colouring must be sufficiently dense. Such a result is impossible however, since the odd numbers provide a counter example and are fairly dense in many senses of the word. Fortunately, this simple parity obstruction disappears in the finite field setting. Indeed, in [4], the following was proved.¹

Theorem: Let p be a prime number, and $A_1, A_2, A_3 \subset \mathbb{F}_p$ be any sets, $|A_1||A_2||A_3| \geq 40p^{\frac{5}{2}}$. Then there are $x, y \in \mathbb{F}_p$ such that $x + y \in A_1$, $xy \in A_2$ and $x \in A_3$.

Now, let $q = p^n$ be an odd prime power and \mathbb{F}_q a finite field of order q . Given a binary linear form $L(X, Y)$ and a binary quadratic form $Q(X, Y)$, define $N_q(L, Q)$ to be the smallest integer k such that for any subset $A \subset \mathbb{F}_q$ with $|A| \geq k$, there exists $(x, y) \in \mathbb{F}_q^2$ with $L(x, y), Q(x, y) \in A$. In this paper we give estimates on the size of $N_q(L, Q)$. Namely, we will prove the following theorem.

Theorem 1: Let \mathbb{F}_q be a finite field of odd order. Let $Q \in \mathbb{F}_q[X, Y]$ be a binary quadratic form with non-zero discriminant and let $L \in \mathbb{F}_q[X, Y]$ be a binary linear form not dividing Q . Then we have

$$\log q \ll N_q(L, Q) \ll \sqrt{q}.$$

This theorem is the content of the next two sections. In the final section, we provide remarks on the analogous problem in the ring of integers modulo N when N is composite.

2 Upper Bounds

Let $L(X, Y)$ be a linear form and $Q(X, Y)$ be a quadratic form, both with coefficients in \mathbb{F}_q . Suppose A is an arbitrary subset of \mathbb{F}_q . We will reduce the problem of solving $L(x, y), Q(x, y) \in A$ to estimating a character sum.

By a multiplicative character, we mean a group homomorphism $\chi : \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$. We say χ is non-trivial if it is not constant, ie. $\chi \not\equiv 1$. We also extend such characters to \mathbb{F}_q with the convention that $\chi(0) = 0$. One of the most useful features of characters is that for χ

¹The author would like to thank J. Solymosi for bringing this result to his attention.

non-trivial, we have

$$\sum_{x \in \mathbb{F}_q} \chi(x) = 0.$$

The quadratic character on \mathbb{F}_q is the character given by

$$\chi(c) = \begin{cases} 1 & \text{if } c \neq 0 \text{ is a square} \\ -1 & \text{if } c \neq 0 \text{ is not a square} \\ 0 & \text{if } c = 0. \end{cases}$$

Lemma 2: Let $Q \in \mathbb{F}_q[X, Y]$ be a binary quadratic form and let $L \in \mathbb{F}_q[X, Y]$ be a binary linear form. Suppose $a, b \in \mathbb{F}_q$. Then there exist $r, s, t \in \mathbb{F}_q$ depending only on L and Q such that

$$|\{(x, y) \in \mathbb{F}_q^2 : L(x, y) = a \text{ and } Q(x, y) = b\}| = |\{y \in \mathbb{F}_q : ry^2 + say + ta^2 = b\}|.$$

Furthermore, $r = 0$ if and only if $L|Q$ and $r = s = 0$ if and only if $L^2|Q$.

Proof. Write $L(X, Y) = a_1X + a_2Y$ where without loss of generality we can assume $a_1 \neq 0$.

We can factor

$$Q(X, Y) = tL(X, Y)^2 + sL(X, Y)Y + rY^2.$$

If $L(x, y) = a$ then we obtain

$$Q(x, y) = ta^2 + say + ry^2.$$

The y^2 coefficient vanishes if and only if $Q = LM$ for some linear form M . The y and y^2 coefficients vanish if and only if $Q = tL^2$. Certainly, any solution to $L(x, y) = a$ and $Q(x, y) = b$ gives a solution y of $ry^2 + say + ta^2 = b$. Conversely, if y is such a solution, setting $x = a_1^{-1}(a - a_2y)$ produces a solution (x, y) . \square

Corollary 3: Let $Q \in \mathbb{F}_q[X, Y]$ be a binary quadratic form and let $L \in \mathbb{F}_q[X, Y]$ be a binary linear form not dividing Q . For $a, b \in \mathbb{F}_q$, the number of solutions to $L(x, y) = a$ and $Q(x, y) = b$ is

$$1 + \chi((s^2 - 4rt)a^2 + 4rb)$$

where χ is the quadratic character.

Proof. The quantity $(sa)^2 - 4r(ta^2 - b)$ is the discriminant of $ry^2 + say + ta^2 - b$. The result follows from the definition of χ and the quadratic formula. \square

In fact, from Lemma 2, we can essentially handle the situation when $L|Q$.

Corollary 4: Let $Q \in \mathbb{F}_q[X, Y]$ be a binary quadratic form and let $L \in \mathbb{F}_q[X, Y]$ be a binary linear form dividing Q . Then $N_q(L, Q) = 1$ if L^2 does not divide Q , otherwise $N_q(L, Q) \geq \frac{q+1}{2}$.

Proof. Let $A \subset \mathbb{F}_q$. The number of pairs (x, y) with $L(x, y), Q(x, y) \in A$ is

$$\sum_{x,y} \mathbf{1}_A(L(x, y)) \mathbf{1}_A(Q(x, y)) = \sum_{a \in A} \sum_{y \in \mathbb{F}_q} \mathbf{1}_A(say + ta^2)$$

by the above lemma. If $sa \neq 0$ then $say + ta^2$ ranges over \mathbb{F}_q as y , and the inner sum is $|A|$. In this case there are in fact $|A|^2$ solutions (x, y) . If $a = 0$ then $0 \in A$ and we can take $(x, y) = (0, 0)$. If $s = 0$ then the sum is $q \sum_{a \in A} \mathbf{1}_A(a^2 t)$. If we set

$$A = \begin{cases} t \cdot N = \{tn : n \in N\} & \text{if } t \neq 0 \\ N & \text{if } t = 0 \end{cases}$$

where N is the set of non-squares in \mathbb{F}_q , then there are no solutions. This shows that $N_q(L, Q) \geq \frac{q+1}{2}$. \square

We now handle the case that L does not divide Q . The following estimate is essentially due to Vinogradov (see for instance the excercises of chapter 6 in [5] for the analogous result for exponentials).

Lemma 5: Let $A, B \subset \mathbb{F}_q$ and suppose χ is a non-trivial multiplicative character. Then if $u, v \in \mathbb{F}_q^\times$

$$\sum_{a \in A} \sum_{b \in B} \chi(ua^2 + vb) \leq 2\sqrt{q|A||B|}.$$

Proof. Let S denote the sum in question. Then

$$|S| \leq \sum_{b \in B} \left| \sum_{a \in A} \chi(ua^2 + vb) \right| \leq |B|^{\frac{1}{2}} \left(\sum_{b \in \mathbb{F}_q} \left| \sum_{a \in A} \chi(ua^2 + vb) \right|^2 \right)^{\frac{1}{2}}$$

by Cauchy's inequality. Expanding the sum in the second factor, we get

$$\begin{aligned} \sum_{a_1, a_2 \in A} \sum_{\substack{b \in \mathbb{F}_q \\ ua_2^2 + vb \neq 0}} \chi \left(\frac{ua_1^2 + vb}{ua_2^2 + vb} \right) &= \sum_{a_1, a_2 \in A} \sum_{\substack{b \in \mathbb{F}_q \\ ua_2^2 + vb \neq 0}} \chi \left(1 + \frac{u(a_1^2 - a_2^2)}{ua_2^2 + vb} \right) \\ &= \sum_{a_1, a_2 \in A} \sum_{b \in \mathbb{F}_q^\times} \chi(1 + u(a_1^2 - a_2^2)b) \end{aligned}$$

after the change of variables $(ua_2^2 + vb)^{-1} \mapsto b$. When $a_1^2 \neq a_2^2$, the values of $1 + u(a_1^2 - a_2^2)b$ range over all values of \mathbb{F}_p save 1 as b traverses \mathbb{F}_q^\times . Hence, in this case, the sum amounts to -1 . It follows that the total is at most $4q|A|$. \square

Recall that the discriminant of a quadratic form $Q(X, Y) = b_1X^2 + b_2XY + b_3Y^2$ is defined to be $\text{disc}(Q) = b_2^2 - 4b_1b_3$.

Proposition 6: Let $Q \in \mathbb{F}_q[X, Y]$ be a binary quadratic form and let $L \in \mathbb{F}_q[X, Y]$ be a binary linear form not dividing Q . Then $N_q(L, Q) \leq 2\sqrt{q} + 1$ if $\text{disc}(Q) \neq 0$ otherwise $N_q(L, Q) \geq \frac{q-1}{2}$.

Proof. Let $A \subset \mathbb{F}_q$. By Corollary 3, the number of pairs (x, y) with $L(x, y), Q(x, y) \in A$ is

$$\sum_{x,y} \mathbf{1}_A(L(x, y)) \mathbf{1}_A(Q(x, y)) = \sum_{a,b \in A} 1 + \chi(Da^2 + 4rb)$$

where $D = s^2 - 4rt$. One can check that in fact $D = a_1^{-2}\text{disc}(Q)$.

If $D = 0$ then $\chi(Da^2 + 4rb) + 1 = \chi(r)\chi(b) + 1$. This will be identically zero if A is chosen to be the squares or non-squares according to the value of $\chi(r)$. Hence, if $\text{disc}(Q) = 0$ then $N_q(L, Q) \geq \frac{q-1}{2}$.

Now assume $D \neq 0$. Summing over $a, b \in A$ the number of solutions is

$$|A|^2 + \sum_{a,b \in A} \chi(Da^2 + 4rb) = |A|^2 + E(A).$$

By Lemma 5, $E(A) < |A|^2$ when $|A| \geq 2\sqrt{q} + 1$ and the result follows. \square

In the case that A has particularly nice structure, we can improve the upper bound. Suppose $q = p$ is prime and A is an interval. Then as above the number of pairs (x, y) with $L(x, y), Q(x, y) \in A$ is

$$|A|^2 + \sum_{a,b \in A} \chi(Da^2 + 4rb).$$

Now

$$\sum_{a,b \in A} \chi(Da^2 + 4rb) \leq \sum_{a \in A} \left| \sum_{b \in A} \chi(Da^2/4r + b) \right|.$$

A well-known result of Burgess states that the inner sum (which is also over an interval) is $o(|A|)$ whenever $|A| \gg p^{\frac{1}{4}+\varepsilon}$ (see [2], chapter 12).

3 A Lower Bound

In this section we give a lower bound for $N_q(L, Q)$ in the case that L does not divide Q and $\text{disc}(Q) \neq 0$. To do so we need to produce a set A such that $L(x, y)$ and $Q(x, y)$ are never both elements of A . Equivalently, we need to produce a set A for which $\chi(Da^2 + 4rb) = -1$ for all pairs $(a, b) \in A \times A$.

Let $a \in \mathbb{F}_q$ and define

$$X_a(b) = \begin{cases} 1 & \text{if } \chi(Da^2 + 4rb) = \chi(Db^2 + 4ra) = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the desired set A will have $X_a(b) = 1$ for $a, b \in A$. The idea behind our argument is probabilistic. Suppose we create a graph Γ with vertex set

$$V = \{a \in \mathbb{F}_q : X_a(a) = 1\}$$

and edge set

$$E = \{\{a, b\} : X_a(b) = X_b(a) = 1\}.$$

These edges appear to be randomly distributed and occur with probability roughly $\frac{1}{4}$. In this setting, $N_q(L, Q)$ is one more than the clique number of Γ (ie. the size of the largest complete subgraph of Γ). Let $G(n, \delta)$ be the graph n vertices that is the result of connecting two vertices randomly and independently with probability δ . Such a graph has clique number roughly $\log n$ (see [1], chapter 10). One is tempted to treat Γ as such a graph and construct a clique by greedily choosing vertices, and indeed this is how the set A is constructed. It is worth mentioning that this model suggests that the right upper bound for $N_q(L, Q)$ is also roughly $\log n$.

Lemma 7: Let $B \subset \mathbb{F}_q$. Then for $a \in \mathbb{F}_q$, we have

$$\sum_{b \in B} X_a(b) = \frac{1}{4} \sum_{b \in B} (1 - \chi(Da^2 + 4rb))(1 - \chi(Db^2 + 4ra)) + O(1).$$

Proof. The summands on the right are

$$(1 - \chi(Da^2 + 4rb))(1 - \chi(Db^2 + 4ra)) = \begin{cases} 4 & \text{if } \chi(Da^2 + 4rb) = \chi(Db^2 + 4ra) = -1 \\ 2 & \text{if } \{\chi(Da^2 + 4rb), \chi(Db^2 + 4ra)\} = \{0, -1\} \\ 1 & \text{if } \chi(Da^2 + 4rb) = \chi(Db^2 + 4ra) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For fixed a , the second and third cases can only occur for $O(1)$ values of b . \square

We will use the following well-known theorem of Weil, see for instance chapter 11 of [2].

Theorem 8 (Weil): Suppose $\chi \in \widehat{\mathbb{F}_q^\times}$ has order $d > 1$ and $f \in \mathbb{F}_q[X]$ is not of the form $f = g^d$ for some $g \in \overline{\mathbb{F}_q}[X]$. If f has m distinct roots in $\overline{\mathbb{F}_q}$ then

$$\left| \sum_{x \in \mathbb{F}_q} \chi(f(x)) \right| \leq m\sqrt{q}.$$

Proposition 9: Let $A, B \subset \mathbb{F}_q$ with $|A|, |B| > \sqrt{q}$. Then

$$\sum_{a \in A} \sum_{b \in B} X_a(b) = \frac{|A||B|}{4} + O(|A||B|^{\frac{1}{2}}q^{\frac{1}{4}}).$$

Proof. By the preceding lemma, it suffices to estimate

$$\begin{aligned} & \sum_{a \in A} \frac{1}{4} \left(\sum_{b \in B} (1 - \chi(Da^2 + 4rb))(1 - \chi(Db^2 + 4ra)) \right) + O(1) \\ &= \frac{|A||B|}{4} - \frac{1}{4} \sum_{a \in A} \sum_{b \in B} \chi(Da^2 + 4rb) - \frac{1}{4} \sum_{a \in A} \sum_{b \in B} \chi(Db^2 + 4ra) \\ & \quad + \frac{1}{4} \sum_{a \in A} \sum_{b \in B} \chi((Da^2 + 4rb)(Db^2 + 4ra)) + O(|A|). \end{aligned}$$

By Lemma 1 of the previous section, the first two sums above are $O(\sqrt{q|A||B|}) = O(|A||B|^{\frac{1}{2}}q^{\frac{1}{4}})$. By Cauchy's inequality, the final sum is bounded by

$$|B|^{\frac{1}{2}} \left(\sum_{b \in \mathbb{F}_q} \left| \sum_{a \in A} \chi((Da^2 + 4rb)(Db^2 + 4ra)) \right|^2 \right)^{\frac{1}{2}}.$$

Expanding the square modulus, the second factor is the square-root of

$$\sum_{a_1, a_2 \in A} \sum_{b \in \mathbb{F}_q} \chi((Da_1^2 + 4rb)(Db^2 + 4ra_1)(Da_2^2 + 4rb)(Db^2 + 4ra_2)).$$

By Weil's theorem, the inner sum is bounded by $6\sqrt{q}$ when the polynomial

$$f(b) = (Da_1^2 + 4rb)(Db^2 + 4ra_1)(Da_2^2 + 4rb)(Db^2 + 4ra_2)$$

is not a square. This happens for all but $O(|A|)$ pairs (a_1, a_2) . Hence the bound is $O(|A|q + |A|^2\sqrt{q})$. Since $|A| > \sqrt{q}$, this is $O(|A|^2\sqrt{q})$ and the overall bound is $O(|A||B|^{\frac{1}{2}}q^{\frac{1}{4}})$. \square

We immediately deduce the following.

Corollary 10: There is an absolute constant $c > 0$ such that if $B \subset \mathbb{F}_q$ with $|B| \geq c\sqrt{q}$ then there is an element $a \in B$ such that

$$|\{b \in B : X_a(b) = 1\}| \geq \frac{1}{8}|B|.$$

Proof. Indeed, taking $A = B$ in the preceding theorem,

$$\max_{a \in B} \left\{ \sum_{b \in B} X_a(b) \right\} \geq \frac{1}{|B|} \sum_{a, b \in B} X_a(b) = \frac{|B|}{4} + O(q^{\frac{1}{4}}|B|^{\frac{1}{2}}) \geq \frac{|B|}{8}$$

when $|B| > c\sqrt{q}$ for some appropriately chosen c . \square

Proposition 11: Let $Q \in \mathbb{F}_q[X, Y]$ be a binary quadratic form and let $L \in \mathbb{F}_q[X, Y]$ be a binary linear form not dividing Q . Then if $\text{disc}(Q) \neq 0$ we have $N_q(L, Q) \gg \log q$.

Proof. We will construct a clique in the graph Γ introduced above. First we claim that $|V| = \frac{q-1}{2} + O(1)$. Indeed

$$\sum_{a \in \mathbb{F}_q^\times} \chi(Da^2 + 4ra) = \sum_{a \in \mathbb{F}_q^\times} \chi(a^{-2})\chi(Da^2 + 4ra) = \sum_{a \in \mathbb{F}_q^\times} \chi(D + 4ra^{-1}) = O(1)$$

by orthogonality. The final term is $O(1)$ and the claim follows since χ takes on the values ± 1 on \mathbb{F}_q^\times .

Now set $V_0 = V$ and assume q is large. Write $|V_0| = c'q > c\sqrt{q}$ (with c as in the preceding corollary and $c' \approx \frac{1}{2}$). For $a \in V_0$, let $N(a)$ denote the neighbours of a (ie. those b which are joined to a by an edge). Then there is an $a_1 \in V_0$ such that $|N(a_1)| \geq c'q/8$. Let $A_1 = \{a_1\}$, let $V_1 = N(a_1) \subset V_0$, and for $a \in V_1$ let $N_1(a) = N(a) \cap V_1$. By choice, all elements of V_1 are connected to a_1 . Now $|V_1 \setminus A_1| \geq c'q/8 - 1 \geq c'q/16$ so, provided this is at least $c'q/16$, there is some element a_2 of $V_1 \setminus A_1$ such that $|N_1(a_2)| \geq |V_1 \setminus A_1|/8$. Let $A_2 = A_1 \cup \{a_2\}$, $V_2 = N_1(a_2) \subset V_1$ and define $N_2(a) = N(a) \cap V_2$. Once again each element of V_2 is connected to each element of A_2 . We repeat this process provided that at stage i there exists an element $a_{i+1} \in V_i \setminus A_i$ with $|N_i(a_{i+1})| \geq |V_i \setminus A_i|/8$. We set $A_{i+1} = A_i \cup \{a_{i+1}\}$ and observe that A_{i+1} induces a clique. We may iterate provided $|V_i \setminus A_i| > c\sqrt{q}$ which is guaranteed for $i \ll \log q$. The final set A_i (which has size i) will be the desired set A . \square

The combination of this proposition and Proposition 6 completes the proof of Theorem 1.

4 Remarks for Composite Modulus

Consider the analogous question in the ring $\mathbb{Z}/N\mathbb{Z}$ with N odd. Let $L(X, Y) = a_1X + a_2Y$ with $(a_1, N) = 1$ and $Q(X, Y) = b_1X^2 + b_2XY + b_3Y^2$. We then let $A \subset \mathbb{Z}/N\mathbb{Z}$ and wish to find $(x, y) \in (\mathbb{Z}/N\mathbb{Z})^2$ such that $L(x, y), Q(x, y) \in A$. As before, this amounts to finding a solution to

$$Q(a_1^{-1}(a - a_2Y), Y) = b$$

for some $a, b \in A$. In general, one cannot find a solution based on the size of A alone unless A is very large. Indeed, if p is a small prime dividing N and $t \bmod p$ is chosen such that the discriminant of

$$Q(a_1^{-1}(t - a_2Y), Y) - t$$

is a non-residue modulo p then taking $A = \{a \bmod N : a \equiv t \bmod p\}$ provides a set of density $1/p$ which fails to admit a solution.

Acknowledgements

I would like to thank Leo Goldmakher and John Friedlander for introducing me to the problem and helpful discussion.

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